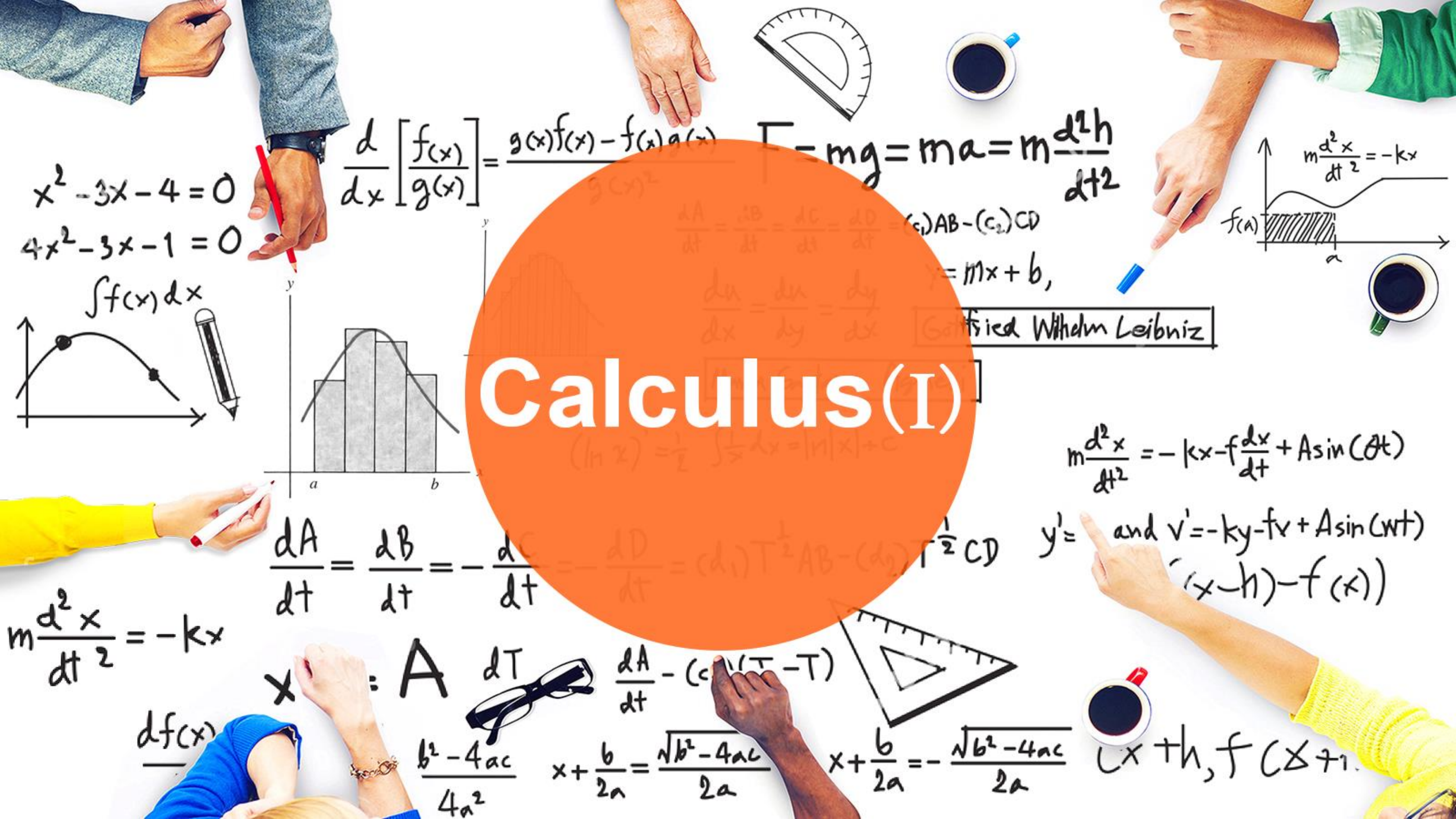
Calculus(I)



x² - 3x - 4 = 0
4x² - 3x - 1 = 0
∫f(x) dx
mg = ma = m d²h/dt²
m d²x/dt² = -kx
m d²x/dt² = -kx - f dx/dt + Asin(θt)
y' = and v' = -ky - fv + Asin(wt)
x + b/2a = √(b² - 4ac)/2a
x + b/2a = - √(b² - 4ac)/2a

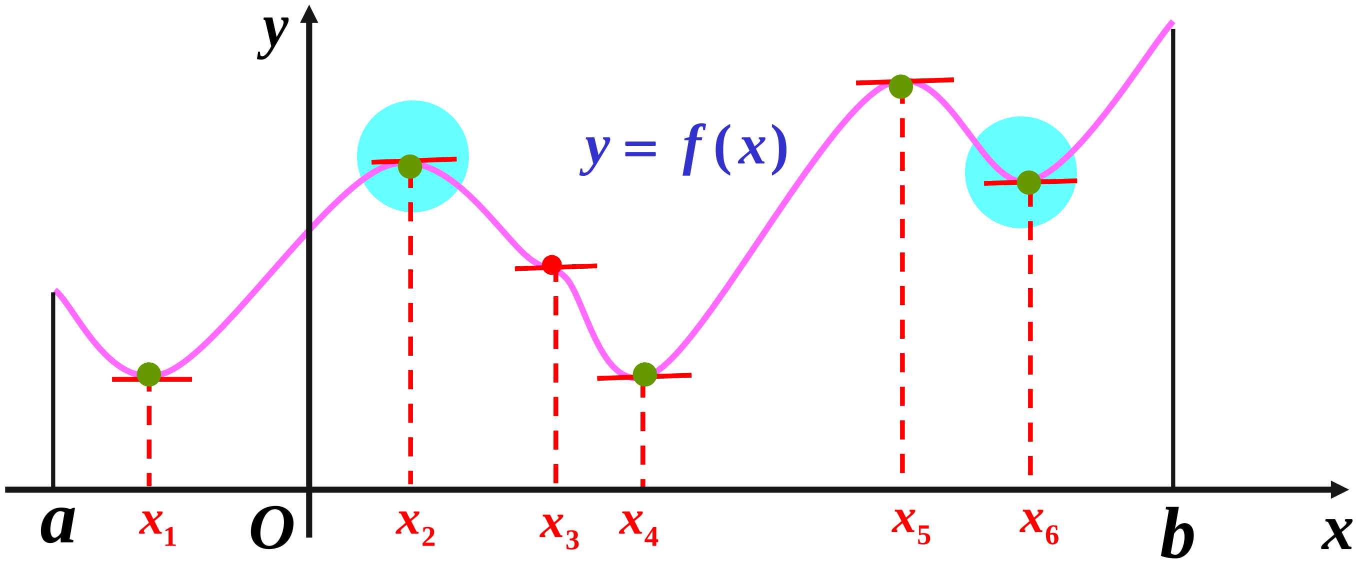
3.3 Local Extrema and Extrema on Open Intervals

Lecturer: Xue Deng

Recall: Global maximum value on S.

? Local maximum value or relative maximum value

For example: What is the local maximum value in the following figure ?



Definition of Local Extrema

Let S, the domain of f, contain the point c.

(i) $f(c)$ is a local maximum value of f if there is an interval (a,b) containing c such that $f(c)$ is the maximum value of f on $(a,b)\cap S$;

(ii) $f(c)$ is a local minimum value of f if there is an interval (a,b) containing c such that $f(c)$ is the minimum value of f on $(a,b)\cap S$;

(iii) $f(c)$ is a local extreme value of f if it is either a local maximum or a local minimum value.

Theorem of Local Extrema

First Derivative Test

Let f be continuous on an open interval (a,b) that contains a critical point c

(i) $f'(x)>0$ for all x in (a,c) and $f'(x)<0$ for all x in (c,b),
then $f(c)$ is a local maximum value of f.(see Figure1)

(ii) $f'(x)<0$ for all x in (a,c) and $f'(x)>0$ for all x in (c,b),
then $f(c)$ is a local minimum value of f.(see Figure2)

(iii) $f'(x)$ has the same signs on both sides of c, then $f(c)$ is not a local extreme value of f.(see Figure3)

Proof of Theorem A: First Derivative Test

Proof of (i): Since $f'(x) > 0$ for all x in (a, c),

f is increasing on $(a, c]$ by the Monotonicity Theorem.

Again, since $f'(x) < 0$ for all x in (c, b),

f is decreasing on $[c, b)$.

Thus, $f(x) < f(c)$ for all x in (a, b), except of course at $x = c$.

We conclude that $f(c)$ is a local maximum.

The proof of (ii) and (iii) are similar.

Local Extrema

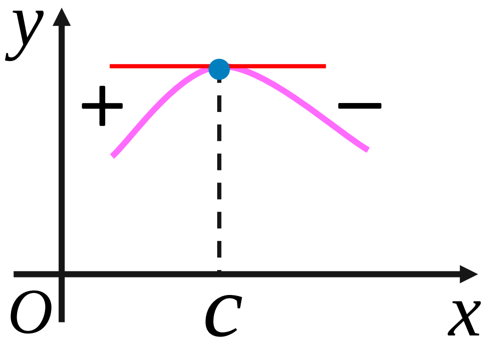


Figure1

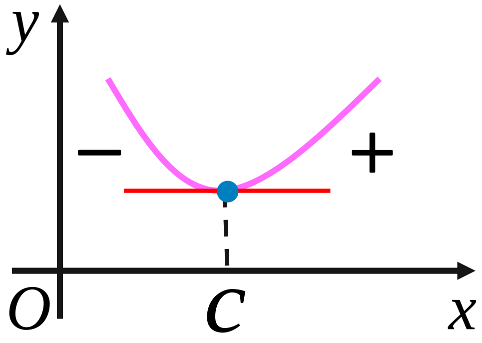


Figure2

a local extrema value

Figure3

not a local extrema value

Theorem of Local Extrema

Second Derivative Test

Let f' and f'' exist at every point in an open interval (a,b) containing c, and suppose that $f'(c)=0$.

(i) If $f''(c)<0$, then $f(c)$ is a local maximum value of f.

(ii) If $f''(c)>0$, then $f(c)$ is a local minimum value of f.

(i) $f''(c)<0 \Rightarrow f''(c)=\lim_{x\to c}\dfrac{f'(x)-f'(c)}{x-c}=\lim_{x\to c}\dfrac{f'(x)-0}{x-c}<0$

$$x\neq c, \frac{f'(x)}{x-c}<0 \Rightarrow f'(x)(x-c)<0 \Rightarrow \begin{cases} x<c & f'(x)>0 \\ x>c & f'(x)<0 \end{cases}$$

$\therefore f(c)$ is a local maximum value.

(ii) Work by yourself.

Example 1

For $f(x) = (x+1)^3 (x-1)^{\frac{2}{3}}$

Use the first Derivative Test to identify local extrema.

(i) $f'(x) = 3(x+1)^2 (x-1)^{\frac{2}{3}} + \frac{2}{3}(x+1)^3 (x-1)^{-\frac{1}{3}}$

$$= \frac{(x+1)^2 (11x-7)}{3(x-1)^{\frac{1}{3}}}$$

(ii) stationary points: $x = -1$, $x = \frac{7}{11}$. singular point: $\boldsymbol{x = 1.}$

(iii) Table

Example 1

(iii) Table

Stationary points: $x = -1, \quad x = \frac{7}{11},$

Singular point: $x = 1.$

$$f'(x) = \frac{(x+1)^2(11x-7)}{3(x-1)^{\frac{1}{3}}}$$

x	$(-\infty,-1)$	-1	$(-1,\frac{7}{11})$	$\frac{7}{11}$	$(\frac{7}{11},1)$	1	$(1,+\infty)$
$f'(x)$	$+$	0	$+$	0	$-$	Not exist	$+$
$f(x)$	↗	not	↗	Local max	↘	Local min	↗

(iv) $f(\frac{7}{11}) \approx 2.2$ is a local maximum value.

$f(1) = 0$ is a local minimum value.

Example 2

For $f(x)=\frac{1}{3}x^3-x^2-3x+4$

Use the Second Derivative Test to identify local extrema.

$$f'(x)=x^2-2x-3=(x+1)(x-3)=0 \Rightarrow x=3, x=-1$$

$$f''(x)=2x-2$$

$$f''(-1)=-2-2=-4<0 \Rightarrow f(-1) \text{ is a local maximum value}$$

$$f''(3)=2\cdot 3-2=4>0 \Rightarrow f(3) \text{ is a local minimum value}$$

Example 3

Find the local extrema value as to $f(x) = x^2 - 6x + 5$ on $(-\infty, +\infty)$

$\because f'(x) = 2(x-3) = 0 \Rightarrow x = 3 (f'(3) = 0)$

$f''(x) = 2 \Rightarrow f''(3) = 2 > 0$

$\therefore f(3)$ is a local minimum value.

Summary

First Derivative Test:

Let f be continuous on an open interval (a,b) that contains a critical point c

(i) $f'(x)>0$ for all x in (a,c) and $f'(x)<0$ for all x in (c,b),

then $f(c)$ is a local maximum value of f.

(ii) $f'(x)<0$ for all x in (a,c) and $f'(x)>0$ for all x in (c,b),

then $f(c)$ is a local minimum value of f.

(iii) $f'(x)$ has the same signs on both sides of c, then $f(c)$ is not a local extreme value of f.

Summary

Second Derivative Test

Let f' and f'' exist at every point in an open interval (a,b) containing c, and suppose that $f'(c)=0$.

(i) If $f''(c) < 0$, then $f(c)$ is a local maximum value of f.

(ii) If $f''(c) > 0$, then $f(c)$ is a local minimum value of f.

Question and Answer

Find the local extrema value as to $f(x) = x^3 + 3x^2 - 24x - 20$.

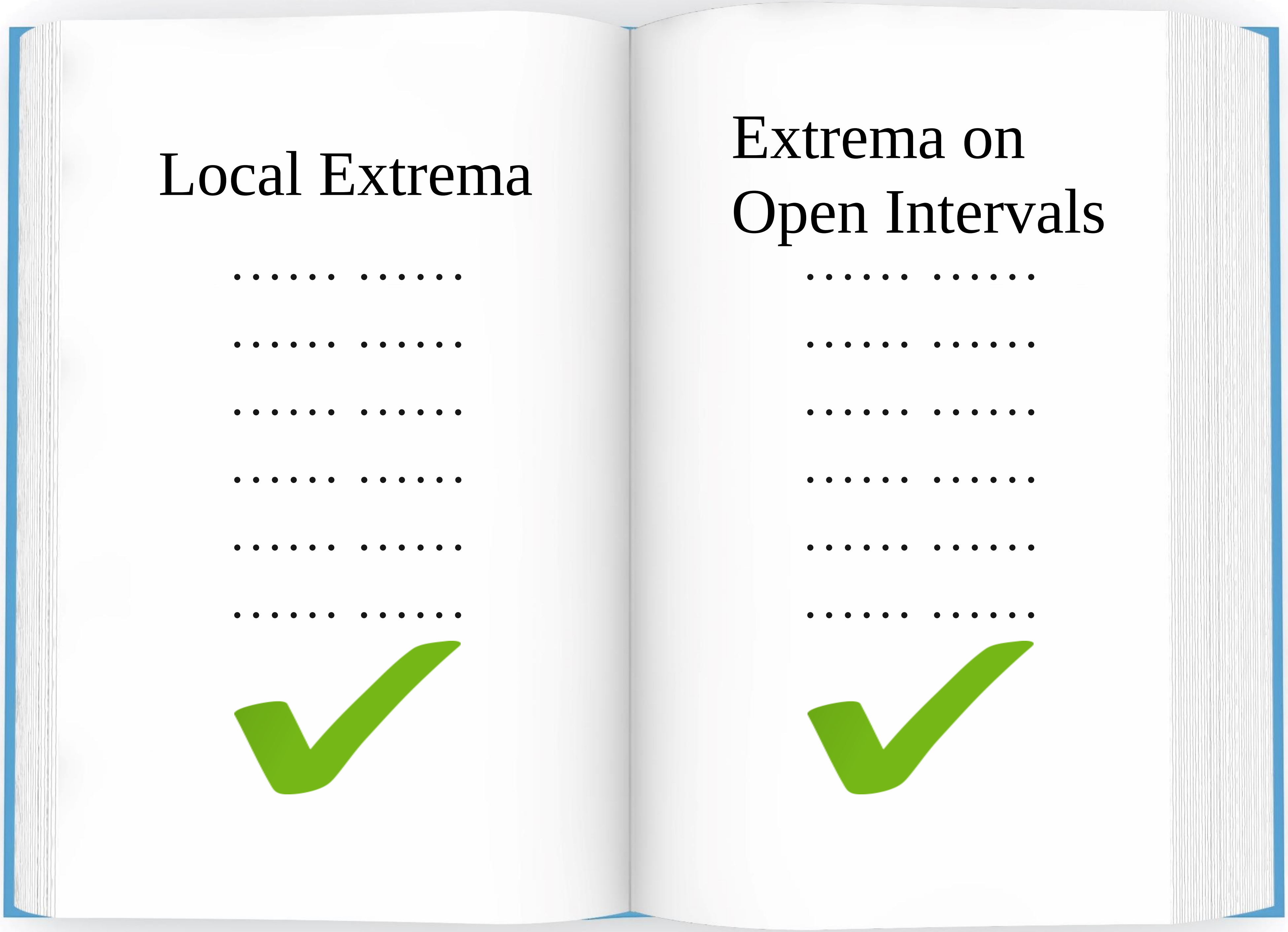
$$f'(x) = 3x^2 + 6x - 24 = 3(x+4)(x-2)$$

Let $f'(x) = 0$, obtain the stationary points $x_1 = -4,\ x_2 = 2.$

$\because\ f''(x) = 6x + 6,$

$\therefore\ f''(-4) = -18 < 0 \Rightarrow$ the local maximum value $f(-4) = 60,$

$f''(2) = 18 > 0 \Rightarrow$ the local minimum value $f(2) = -48.$


Local Extrema
Extrema on Open Intervals